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Paper V
Topic Differential operators in terms of orthogonal curvilinear coordinates
Gradient, Divergence

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Differential Operators in Terms of "Orthogonal Curvilinear Coordinates"

Consider three mutually perpendicular coordinate surfaces described by q_1 = constant, q_2 = constant, q_3 = constant. Let $\psi(q_1, q_2, q_3)$ be a scalar fun^n and V be a vector fun^n with components V_1, V_2, V_3 in the three directions in which q_1, q_2, q_3 increase. If $\hat{u}_1, \hat{u}_2, \hat{u}_3$ are unit vectors along the direction of increasing q_1, q_2, q_3 respectively then vector V in terms of orthogonal curvilinear coordinates may be written as

$$V = \hat{u}_1 V_1 + \hat{u}_2 V_2 + \hat{u}_3 V_3 \quad \text{—— (11)}$$

(I) **Gradient** :— The gradient of a scalar $fun^n \psi$ is a vector whose magnitude and direction give the maximum space rate of change of scalar $fun^n \psi$. From this interpretation the component of $\nabla\psi(q_1, q_2, q_3)$ in the direction normal to the surface q_1 = constant and hence in the direction of q_1 is

$$\nabla\psi\Big|_1 = \lim_{\delta s_1 \to 0} \frac{\delta \psi_1}{\delta s_1} = \frac{\partial \psi}{\partial s_1} = \frac{\partial \psi}{h_1 \partial q_1} = \frac{1}{h_1}\frac{\partial \psi}{\partial q_1} \quad \text{—— (12)}$$

where $\delta s_1 = h_1 \delta q_1$ is the differential length in the direction of increasing q_1 and $\partial\psi$ represents an

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increase in ψ on travelling a distance δs_1 in the limit $\delta s_1 \to 0$

By repeating eqn (12) for q_2 and again for q_3, we get

$$\nabla\psi\Big|_2 = \frac{1}{h_2}\frac{\delta\psi}{\partial q_2} \quad \text{— (13)}$$

$$\nabla\psi\Big|_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial q_3} \quad \text{— (14)}$$

Adding eqns (12), (13) and (14) vectorially, the gradient of scalar funn ψ in orthogonal curvilinear coordinates becomes

$$\text{grad}\,\psi = \nabla\psi = \frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{u}_3}{h_3}\frac{\partial\psi}{\partial q_3} \quad \text{— (15)}$$

Thus the operator grad in orthogonal curvilinear coordinates is

$$\text{grad} = \nabla = \frac{\hat{u}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{u}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{u}_3}{h_3}\frac{\partial}{\partial q_3} \quad \text{— (16)}$$

(2) **Divergence** :— The divergence of a vector V is written as

$$\text{div}\, V = \nabla\cdot V = \nabla\cdot\left(\hat{u}_1 V_1 + \hat{u}_2 V_2 + \hat{u}_3 V_3\right)$$

$$= \nabla\cdot\left(\hat{u}_1 V_1\right) + \nabla\cdot\left(\hat{u}_2 V_2\right) + \nabla\cdot\left(\hat{u}_3 V_3\right) \quad \text{— (17)}$$

from ~~section~~ orthogonal curvilinear coordinates, we have

$$\text{div}\left(\hat{u}_1 V_1\right) = \nabla\cdot\left(\hat{u}_i V_i\right) = V_i\,\nabla\cdot\hat{u}_i + \hat{u}_i\cdot\nabla V_i \quad \text{— (18)}$$

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Thus, in order to find $\nabla \cdot V$, it is needed to obtain $\nabla \cdot \hat{u}_i$ which may be obtained as follows:

Remembering that $\frac{\hat{u}_1}{h_1}$ is a product of a scalar and a vector,

we may write in view of relation

$$\text{Curl}(\phi \vec{A}) = \phi \, \text{curl} \, \vec{A} + \text{grad} \, \phi \times \vec{A}$$

$$\text{Curl}\left(\frac{\hat{u}_1}{h_1}\right) = \nabla \times \left(\frac{\hat{u}_1}{h_1}\right) = \nabla\left(\frac{1}{h_1}\right) \times \hat{u}_1 + \frac{1}{h_1}(\nabla \times \hat{u}_1) \quad \text{— (19)}$$

Using eqn (16), we see that

$$\nabla q_1 = \frac{u_1}{h_1}.$$

Since curl grad $q_1 = 0$, i.e., $\nabla \times \frac{\hat{u}_1}{h_1} = 0$, therefore eqn (19) yields

$$\hat{u}_1 \times \nabla\left(\frac{1}{h_1}\right) = \frac{1}{h_1}(\nabla \times \hat{u}_1) \quad \text{— (20)}$$

Using eqn (16) again and performing the differentiation, we obtain

$$\nabla\left(\frac{1}{h_1}\right) = \frac{\hat{u_1}}{h_1^2 h_1}\frac{\partial h_1}{\partial q_1} - \frac{\hat{u_2}}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{\hat{u_3}}{h_1 h_3}\frac{\partial h_1}{\partial q_3} \quad \text{--- (21)}$$

Using this eqn (20) gives

$$\nabla \times \hat{u}_1 = -\frac{\hat{u}_1 \times \hat{u}_1}{h_1^2}\frac{\partial}{\partial q_1} - \frac{\hat{u}_1 \times \hat{u_2}}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{u_1 \times \hat{u}_3}{h_1 h_3}\frac{\partial h_1}{\partial q_3} \quad \text{--- (22)}$$

Let us now choose the positive direction of $\hat{u}_1, \hat{u}_2, \hat{u}_3$ so that (q_1, q_2, q_3) or $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ from a right handed set. Then recalling that $\hat{u}_i \times \hat{u}_i = 0$, $\hat{u}_i \cdot \hat{u}_j = 0$.

$\hat{u}_i \times \hat{u}_j = \hat{u}_k$, $\hat{u}_i \cdot \hat{u}_i = 1$,

In particular $\hat{u}_1 \times \hat{u}_1 = 0$, $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$ and $\hat{u}_1 \times \hat{u}_3 = -\hat{u}_2$. Substituting these values in eqn (22), we obtain

$$\nabla \times \hat{u}_1 = \frac{\hat{u}_2}{h_1 h_2}\frac{\partial h_1}{\partial q_3} - \frac{\hat{u}_3}{h_1 h_2}\frac{\partial h_1}{\partial q_2} \quad \text{--- (23a)}$$

Now by cyclic permutation of coordinates, similar results for curl of $\hat{u}_2$ and $\hat{u}_3$ may be written i.e.,

$$\nabla \times \hat{u}_2 = \frac{\hat{u}_3}{h_2 h_1}\frac{\partial h_2}{\partial q_1} - \frac{\hat{u}_1}{h_2 h_3}\frac{\partial h_2}{\partial q_3} \quad \text{--- (23b)}$$

$$\nabla \times \hat{u}_3 = \frac{\hat{u}_1}{h_3 h_2}\frac{\partial h_2}{\partial q_2} - \frac{\hat{u}_2}{h_3 h_1}\frac{\partial h_3}{\partial q_1} \quad \text{--- (23c)}$$

ations. The probability of thermodynamics fluctuations.

Keeping in mind the relation
$div\,(a \times b) = b \cdot curl\, a - a \cdot curl\, b$,

we have

$$\nabla \cdot \hat{u}_1 = \nabla \cdot (\hat{u}_2 \times \hat{u}_3) = \hat{u}_3 \cdot (\nabla \times \hat{u}_2) - \hat{u}_2 \cdot (\nabla \times \hat{u}_3) \quad \text{—— (24)}$$

Substituting the values of $\nabla \times \hat{u}_2$ and $\nabla \times \hat{u}_3$ from (23) and the fact that $\hat{u}_i \cdot \hat{u}_i = 1$, $\hat{u}_i \cdot \hat{u}_j = 0$, equ^n (24) yields

$$\nabla \cdot \hat{u}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial (h_2 h_3)}{\partial q_1} \quad \text{—— (25)}$$

From (16), (18) and (25) we get

$$\nabla \cdot (\hat{u}_1 V_1) = V_1 \nabla \cdot \hat{u}_1 + \hat{u}_1 \cdot \nabla V_1$$

$$= \frac{V_1}{h_1 h_2 h_3} \frac{\partial (h_2 h_3)}{\partial q_1} + \hat{u}_i \cdot \left(\frac{\hat{u}_1}{h_1} \frac{\partial V_1}{\partial q_1} + \frac{\hat{u}_2}{h_2} \frac{\partial V_1}{\partial q_2} + \frac{\hat{u}_3}{h_3} \frac{\partial V_1}{\partial q_3} \right)$$

$$= \frac{V_1}{h_1 h_2 h_3} \frac{\partial (h_2 h_3)}{\partial q_1} + \frac{1}{h_1} \frac{\partial V_1}{\partial q_1} = \frac{1}{h_1 h_2 h_3} \frac{\partial (V_1 h_2 h_3)}{\partial q_1}$$

—— (26)

Again by cyclic permutation of coordinates

$$\nabla \cdot (\hat{u}_2 V_2) = \frac{1}{h_1 h_2 h_3} \frac{\partial (V_3 h_3 h_1)}{\partial q_2} \quad \text{— (26b)}$$

$$\nabla \cdot (\hat{u}_3 \cdot V_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial (V_3 h_1 h_2)}{\partial q_3} \quad \text{— (26c)}$$

Using eqn (26), eqn (17) reduces to

$$\text{div } V = \nabla \cdot V = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (V_1 h_2 h_3)}{\partial q_1} + \frac{\partial (V_2 h_3 h_1)}{\partial q_2} + \frac{\partial (V_3 h_1 h_2)}{\partial q_3} \right] \quad \text{— (27)}$$

This eqn represents div V in orthogonal curvilinear coordinates.